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# Relaxation to equilibrium and the spacing distribution of zeros of the Riemann $\zeta$ function

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**Abstract**

A novel approach is proposed to the distribution of spacings between zeros of the Riemann zeta function. Starting from the observation that the spacing distribution for zeros near the real axis is sharper than the asymptotic distribution, and that all computed moments grow monotonically as zeros are computed farther and farther away from $\Im(z) = 0$, an analogy with relaxation to equilibrium in a statistical system is drawn. Namely, it is conjectured that the spacing distribution evolves in a fictitious time $t = \sqrt{\log_{10} T}$, where $T$ is the zeros' imaginary part, in the same way as the eigenvalues of a random matrix with mixed GUE and GSE symmetries. The time evolution restores asymptotically (at $T \to \infty$) the GUE symmetry. Dyson's Brownian motion model for the eigenvalues of a random matrix is used for describing the time evolution, and an approximate, analytic description of the spacing distribution is conjectured, valid to first order in $\exp(-t)$.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

If it is true that mathematics is the language of physics, it may happen that physics hold the key to a difficult mathematical problem. One of these rare instances seems to be the proof of Riemann's conjecture. The connection between physics and mathematics is provided in this case by random matrix theory (RMT) [1], a statistical theory devised for describing the fluctuation properties of complex nuclear spectra, and that has found an application in several fields of physics.

Riemann's conjecture states that the nontrivial zeros of the so-called Riemann zeta function, $\zeta(z)$, are all of the form $z_n = 1/2 + \mathrm{i}T_n$ [2]. In pursuing a proof of the conjecture, billions of zeros have been computed numerically [3]. Besides lending support to the conjecture, knowledge of the zeros allows one to investigate their statistical properties, as

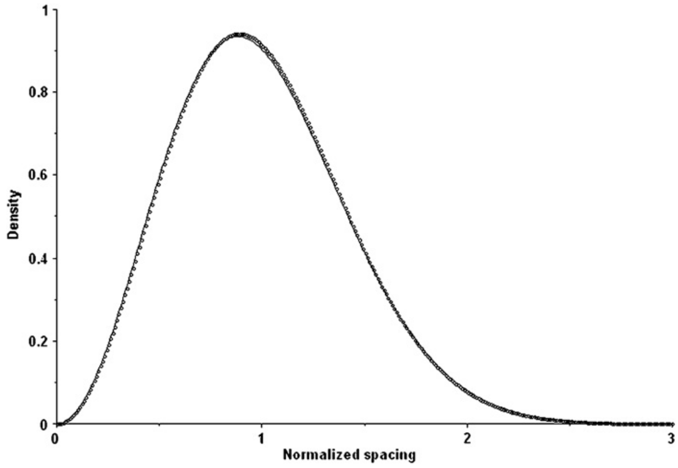


**Figure 1.** The distribution of zero spacings (open circles) plotted together with the asymptotic GUE distribution of consecutive eigenvalue spacings (solid line).

a function of the 'height' $T$ along the imaginary axis $\Re(z) = 1/2$. In a tea-time discussion with F Dyson in Princeton, H Montgomery learned that the zeros share the asymptotic, $T \to \infty$ form of their pair correlation function, with the $N \to \infty$ limit of the eigenvalue correlation function of $N \times N$ unitary random matrices [4]. Note that, although mathematicians usually prefer referring to the circular ensemble of unitary matrices (CUE), I will follow here the use, more common among physicists, of the notation GUE, that refers to the Gaussian unitary ensemble. The asymptotic pair correlation function of eigenvalues is identical for CUE and GUE matrices [1].

Following Montgomery's conjecture, the relations between the statistical properties of the zeros of the zeta function and those of the eigenvalues of random matrices have been intensely studied. For instance, figure 1 shows the distribution of the spacings between consecutive zeros computed around $T = 10^{16}$ [3], rescaled in such a way as to have unit mean. The solid line is the prediction from RMT for unitary random matrices (GUE). The agreement is visually excellent. However, a non-vanishing difference exists between the asymptotic, $N \to \infty$ GUE distribution of RMT and the finite-$T$ result for the zeros. As Odlyzko discovered numerically [5], such a difference has a precise structure, which one should be able to predict.

This communication addresses such a prediction from physical and computational perspectives. It will be argued that the distribution of zero spacings (DZS) tends to its asymptotic form in a way which is described by Dyson's Brownian motion model [6], and which is indeed akin to relaxation to equilibrium in statistical physics. I shall show numerically that such relaxation takes place as a function of a fictitious time $t = (\log_{10} T)^{1/2}$. As will be appreciated, the present argument is entirely heuristic—as a physicist, I would say phenomenological. It is based on an approximate treatment of the spacing distribution, which fits numerical results. Nevertheless, the numerical evidence discussed here shows that the use of Dyson's Brownian motion model for capturing the corrections to asymptotics of the zeros of Riemann zeta function is well founded, and very promising.

## 2. The DZS at finite *T*: corrections to asymptotic behaviour

An extensive numerical study of the statistical properties of zeros 'high up' along the $T$-axis has been reported on his website by Xavier Gourdon [7], who computed the first $10^{13}$ zeros,

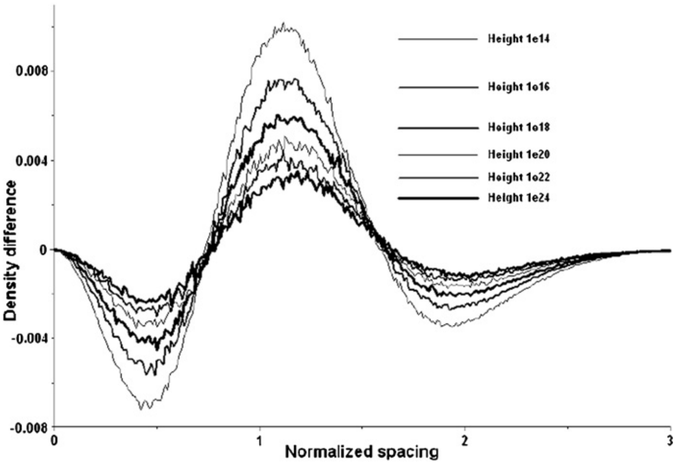


**Figure 2.** The difference between the zero spacing distribution $P_\zeta(s, T)$ and the asymptotic GUE distribution, at various $T$ (from [7]).

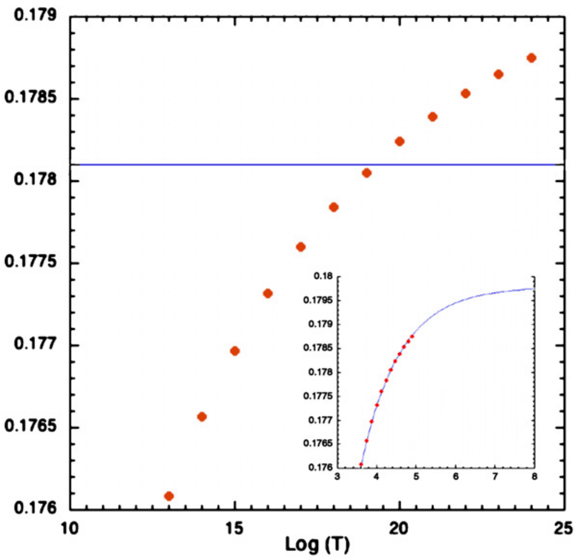


**Figure 3.** The values of the variance $\sigma^2(T)$ of $P_\zeta(s, T)$ as computed in [7] (solid circles), plotted as a function of $\log_{10}(T)$. The horizontal straight line marks the value $\sigma_2^2 = 0.1781$ of the variance of the Wigner surmise for $\beta = 2$. Inset: the variance $\sigma^2(T)$ plotted as a function of $\sqrt{\log_{10}(T)}$. The thin continuous curve (blue online) is a fit to equation (3).

and then investigated the DZS for values of the imaginary part $T$ between $10^{13}$ and $10^{24}$. I will use two results of that study: the first one is the graph of the differences between the DZS at various values of $T$, $P_\zeta(s, T)$, and the asymptotic GUE distribution, $P_{\mathrm{GUE}}(s)$. The variable $s$ is the spacing divided by its $T$-dependent mean value. The second result I will use is the list of the computed values of the variance of $P_\zeta(s, T)$ as a function of $T$. Figure 2, taken from [7], shows the evolution of $P_\zeta(s, T) - P_{\mathrm{GUE}}(s)$ at different $T$s. Figure 3 depicts the variance of the DZS as a function of $\log_{10} T$ (for typographical convenience). For future use, the horizontal straight line in figure 3 marks the variance of the Wigner surmise. The latter is the exact distribution of the eigenvalues of a random $2 \times 2$ unitary matrix and it is known to be an excellent approximation for $P_{\text{GUE}}(s)$, as well. As a reminder, the Wigner surmise $P_2(s)$ for random unitary matrices has the form

$$P_2(s) = (32/\pi^2)s^2 \exp(-4s^2/\pi), \tag{1}$$

and its variance $\sigma_2^2 = 3\pi/8 - 1 \approx 0.17810$. Subscript 2 refers to the parameter, usually called $\beta$ in RMT literature, labelling the three Gaussian ensembles: $\beta = 1$ for orthogonal, $\beta = 2$ for unitary and $\beta = 4$ for symplectic matrices, respectively [1].

It is clear from figures 2 and 3 that at small $T$, $P_\zeta(s, T)$ is rather sharp, narrower and higher than the asymptotic $P_{\text{GUE}}(s)$, the maximum of $P_\zeta(s, T)$ decreasing and its width increasing as zeros of larger imaginary part are computed.

In a recent paper [8], Bogomolny *et al* have ascribed the difference between the spacing distribution at finite $T$ and its asymptotic form to a finite-$N$ effect: the finite-$T$ $P_\zeta(s, T)$ would thus be the level distribution of some finite-$N$ matrix, and the asymptotic form would be obtained as $N$ goes to $\infty$. In fact, Bogomolny *et al* have argued heuristically that a given 'height' $T$ above the real axis corresponds to an effective matrix dimension $N_{\text{eff}} = \ln(T/2\pi)/(\sqrt{12\Lambda})$, where $\Lambda = 1.57314\ldots$ is a well-defined constant [8]. I shall discuss below what I believe to be a major drawback of this approach, i.e. its inconsistency with exact results.

An alternative attitude may consist in assuming that the change of $P_\zeta(s, T)$ with $T$ follows that of random matrices whose symmetry is broken at $T = 0$, and restored as $T \to \infty$. Specifically, $P_\zeta(s, T)$ at small $T$ may correspond to the spacing distribution of eigenvalues of GUE random matrices whose symmetry is broken by a symplectic (GSE) perturbation. The imaginary part of the zeros, $T$, plays thus the role of the 'strength' of the GUE component, so that when $T \to \infty$ the GUE symmetry is fully restored. According to RMT, the 'height' $T$ on the $\Re(z) = 1/2$ axis, or some function of it, might then be interpreted as an effective 'time'; in this spirit, computing zeros of increasing $T$ would correspond to a time evolution towards equilibrium; and the asymptotic GUE would be the form of the equilibrium distribution. The restoring of a broken symmetry is a phenomenon which has long been studied in the framework of RMT [1, 9]. The Brownian motion model of Dyson is one of the most useful tools for describing such an effect.

## 3. Dyson's Brownian motion model

When constructing RMT, Dyson discovered that the well-known algebraic repulsion between eigenvalues that characterizes the spectrum of random matrices could be interpreted as coming from a logarithmic interaction between particles, the latter being the eigenvalues themselves [1, 6, 9]. Indeed, Dyson realized that the eigenvalue distribution of random matrices is the time-independent solution of a multi-dimensional Fokker–Planck equation, associated with the Langevin equations

$$\dot{x}_m = -2\beta x_m + \beta \sum_{i>j} \frac{1}{x_i - x_j} + \eta \tag{2}$$

where $x_m$ are the eigenvalues (particle positions), $\beta = 1, 2$ or 4 is the parameter characterizing the symmetry of the random matrix and $\eta$ is a Gaussian distributed white noise. Incidentally, note that the values 1, 2 and 4 of $\beta$ do not play any special role in Dyson's model.

One of the consequences of Dyson's Brownian motion model is that, once $\beta$ is fixed in (2), the distribution of the eigenvalues tends to a stationary form corresponding to that $\beta$,

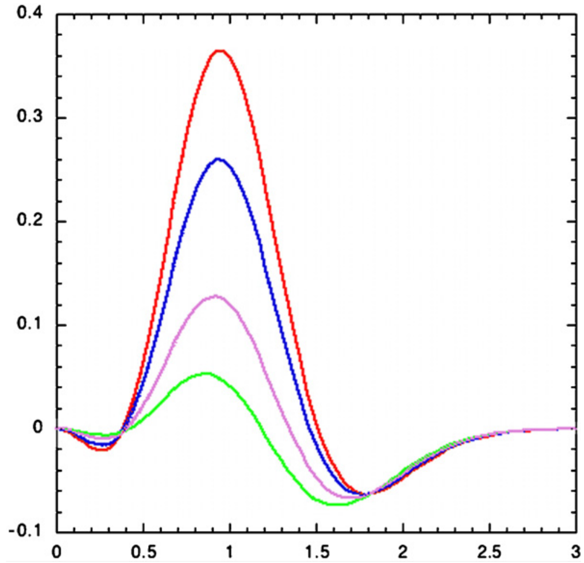


**Figure 4.** The difference between the mean-field solution of Dyson's Brownian motion model for $\beta = 2$, starting from a delta-function distribution at $t = 0$, and the asymptotic Wigner surmise, at increasing time $t$. $t$ increases from top curve (red online) to the lowest one (green online). The solution is explicitly given in [10].

independently of the initial state [6, 9]. Thus, if $\beta$ in equation (2) is fixed, say $\beta = 2$, and that the initial distribution corresponds to, say, $\beta = 4$, then the eigenvalues evolve in time under equation (2), and their distribution changes from $\beta = 4$ to $\beta = 2$.

One of the few exact results known for such transitions is that the variance of the distribution varies in time as [6]

$$\sigma^2(t) = \sigma_0^2 \exp(-t) + \sigma_\infty^2[1 - \exp(-t)]. \tag{3}$$

The inset in figure 3 shows a fit to equation (3) of the variance $\sigma^2(t)$ of $P_\zeta(s, T)$ at various 'times' $t$, assuming $t = (\log_{10} T)^{1/2}$. The latter choice will be discussed below. The fit is very good, and is a prima facie evidence for the relevance of the Brownian motion approach.

Exact results for the time evolution in the context of Dyson's model have been derived for $2 \times 2$ matrices [9], as well as for a mean-field variant of equation (2), which is amenable to an exact solution $P_{MF}(s, t)$ for any time $t$ [10]. The asymptotic, equilibrium solution is in both cases the Wigner surmise $P_\beta(s)$.

As an example, figure 4 exhibits the difference between the solution of the mean-field model for $\beta = 2$ and its asymptotic Wigner distribution $P_2(s)$ at different times, when the initial condition is a delta function centred in $s = 1$. In this context, a delta function may be thought of as a Wigner distribution $P_\beta(s)$ with $\beta \to \infty$. One can see that the qualitative behaviour looks very similar to that of figure 2. An explicit solution has also been found for the transition between $\beta = 4$ and $\beta = 2$: it is given in [10], and it contains a Kummer hypergeometric function which makes it very bad-looking. I prefer to plot in figure 5 the explicit approximating form

$$\delta P_{4\to 2}(s, t) \equiv P_{MF}(s, t) - P_2(s) \approx [P_4(s) - P_2(s)] \exp(-t). \tag{4}$$

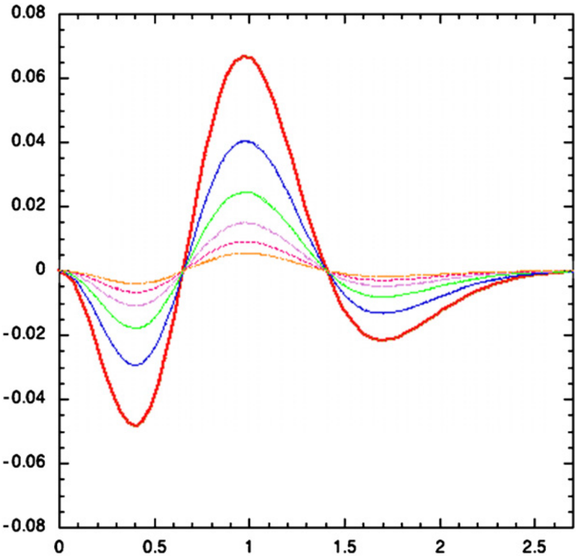


**Figure 5.** A plot of equation (4) at different times, increasing from top curve (red online) to lowest one (orange online).

The right-hand side is indeed an interpolating function that has the merit of giving equation (3) exactly, since of course $P_{MF}(s,t) \approx P_4(s)\exp(-t) + P_2(s)[1-\exp(-t)]$, so that $\sigma^2_{MF}(t) = \sigma^2_4 \exp(-t) + \sigma^2_2[1-\exp(-t)]$.

It can easily be appreciated that the resemblance between figures 3 and 5 is very strong.

## 4. A conjecture: the DSZ as a result of a symmetry-restoring transition

I will now make my conjecture explicit: the zeros of the Riemann zeta function are distributed as the eigenvalues of random matrices of GUE symmetry, submitted to a symmetry-breaking GSE perturbation:

$$P_\zeta(s, t(T)) = P_{\text{GSE+GUE}\to\text{GUE}}(s, t(T)). \tag{5}$$

The large-$T$ limit corresponds to restoring the GUE symmetry, according to Dyson's Brownian motion model.

Starting from this assumption, I will heuristically derive an explicit, analytical approximation to the DZS $P_\zeta(s,t)$ as a function of $t = t(T)$. This approximation will reproduce the behaviour of figure 3, and yield a parameter-free fit to the moments of the DZS.

A further assumption concerns the fictitious time $t$. It is commonly conjectured in number theory that the relation $N = \ln(T/2\pi)$ between matrix size $N$ and imaginary part $T$, holds. In Dyson's model, symmetry is restored when the typical time scale of the fictitious time evolution is $t \sim \sqrt{N}$. We are then led to assume $t \sim \sqrt{\ln T}$. In fact, the numerical evidence points out that base-10 logarithms are the natural choice (although of course base-$e$ logs are natural logs)! We conjecture then that the fictitious time is

$$t = \sqrt{\log_{10} T} \tag{6}$$

as far as zeros of the Riemann zeta function are concerned. I have not yet been able to justify this numerical result.

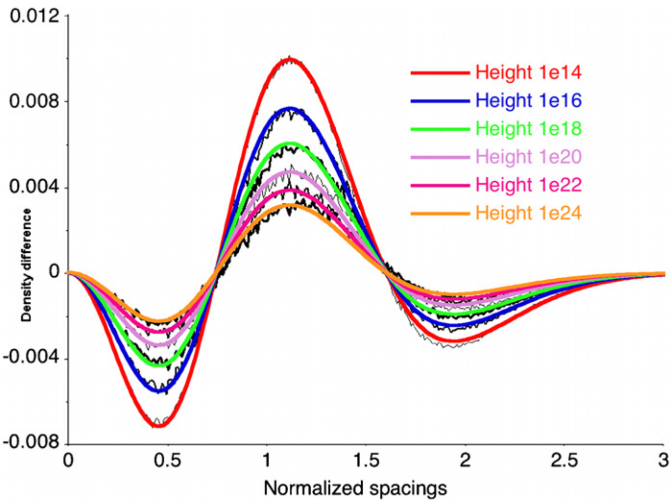


**Figure 6.** Equation (8) compared to the difference between the zero spacing distribution $P_\zeta(s, T)$ and the asymptotic GUE distribution at various $T$ as in figure 2 (see the text for details) (colour online).

I would expect that the exact zero distribution, $P_\zeta(s, t)$, alias $P_{\mathrm{GSE+GUE\to GUE}}(s, t(T))$, can be expanded near $T \to \infty$ in powers of $\exp(-t)$ about the mean-field distribution:

$$\begin{aligned} P_\zeta(s, t(T)) &= P_{\mathrm{GSE+GUE\to GUE}}(s, t(T)) = P_{MF}(s, t) + O(\exp(-2t)) \\ &\approx P_2(s) + \kappa\{[P_4(s) - P_2(s)]\exp(-t)\} + O(\exp(-2t)) \end{aligned} \tag{7}$$

where equation (4) has been used. Higher-order contributions, following Bogomolny *et al* [8], can be included by a shift in the $s$ variable, $s \to \alpha s$, with $\alpha$ being in principle time-dependent. As a first approximation, I will take $\alpha$ a constant. Figure 6 exhibits the fit of $\delta P_\zeta(s, t) = P_\zeta(s, t) - P_{\mathrm{GUE}}(s)$ (data in figure 2) using the expression

$$\delta P_\zeta(s, t) = \kappa[P_4(\alpha s) - P_2(\alpha s)]\exp(-t)]. \tag{8}$$

The fit in the figure yields $\kappa = 1.418$, $\alpha = 0.88$ and $t = \sqrt{\log_{10} T}$. The agreement is striking, considering that just two free parameters are employed. An even better agreement is obtained by making $\alpha$ time-dependent. Heuristically again, I find that

$$\alpha(t) = \exp(-t/29) \tag{9}$$

yields a very close match to all the numerical data.

But we can try to do more than that. The Wigner surmise for $\beta = 2$, $P_2(s)$, is the asymptotic solution of equation (2) in mean field. As such, the actual level distribution for random unitary matrices, $P_{\mathrm{GUE}}(s)$, may be written as

$$P_{\mathrm{GUE}}(s) = P_2(s) + \delta P_{\mathrm{GUE}}(s). \tag{10}$$

Looking at figure 2, one sees that the variance of $P_\zeta(s, t)$ attains the value of the one of the Wigner surmise at $T$ about $2 \times 10^{19}$. Thus, I expect that $P_\zeta(s, t) = P_2(s)$ at this height. In other words, I expect that

$$P_\zeta(s, t) = P_2(s) + \delta P_{\mathrm{GUE}}(s)\exp(-t_0) + \kappa[P_4(\alpha s) - P_2(\alpha s)]\exp(-t), \tag{11}$$

with $\delta P_{\mathrm{GUE}}(s)$ and $t_0$ such that

$$\delta P_{\mathrm{GUE}}(s)\exp(-t_0) = -\kappa[P_4(\alpha s) - P_2(\alpha s)]\exp(-t_0). \tag{12}$$

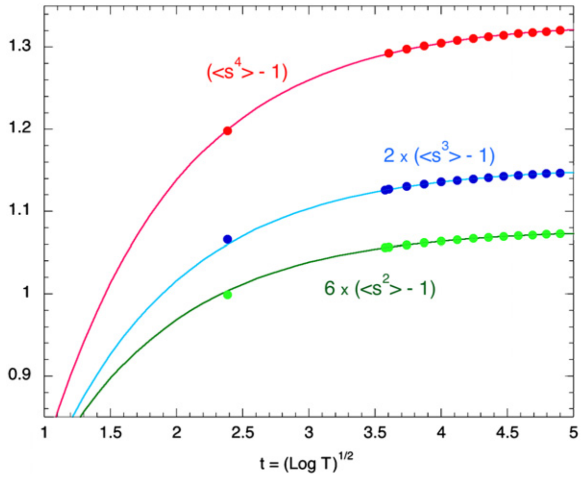


**Figure 7.** A comparison of the first three non-trivial moments of $P_\zeta(s, T)$ computed with the approximate form (14), with the values computed in [7] (colour online).

An inspection of figure 4.2 of Haake's book [9] shows that $\delta P_{\mathrm{GUE}}(s)$ has indeed the same general shape as $\delta P_\zeta(s, t)$ for a given $t$.

As a result, we have an analytic approximation to the whole distribution of zero spacing for the Riemann zeta function, valid at all $T$:

$$P_\zeta(s, t) = P_2(s) + \kappa'\{P_2(\alpha s) - P_4(\alpha s)][1 - \exp(-[t - t_0])\}, \tag{13}$$

where $\kappa' = \kappa \exp(-t_0)$, $t = \sqrt{\log_{10} T}$.

This analytic expression allows us to compute explicitly the various moments of $P_\zeta(s, t)$ as a function of $t(T)$. They are immediately available from the moments of the Wigner surmise for $\beta = 2$ and 4. The general expression of the $n$th moment as a function of $t$ reads:

$$\begin{aligned}\langle s^n\rangle_\zeta(t) &= \langle s^n\rangle_2 + \frac{\kappa'}{\alpha^{n+1}}\{(\langle s^n\rangle_2 - \langle s^n\rangle_4][1 - \exp(-[t - t_0])\} \\ &= \left(\frac{\pi}{4}\right)^{\frac{n-1}{2}} \Gamma\left(\frac{3+n}{2}\right) + \frac{\kappa'}{\alpha^{n+1}}\left[\left(\frac{\pi}{4}\right)^{\frac{n-1}{2}} \Gamma\left(\frac{3+n}{2}\right)\right. \\ &\quad \left. - 2^{-n}\left(\frac{9\pi}{16}\right)^{\frac{n-1}{2}} \Gamma\left(\frac{5+n}{2}\right)\right][1 - \exp(-[t - t_0])].\end{aligned} \tag{14}$$

The second, third and fourth moments are plotted in figure 7, with $\alpha = 0.884\,56$, $\kappa' = 0.016$ and $t_0 = 4.38$. The computed values are compared with the results of Gourdon: once more, the agreement is very satisfactory, even though no automatic best-fitting procedure is involved.

## 5. Discussion and conclusions

I mentioned that an alternative, more orthodox approach has been taken by Bogomolny *et al* [8]. They assume that corrections to the asymptotics must be looked for in the difference between the universal ($N \to \infty$) and the finite-$N$ distribution of levels of random matrices. I will briefly summarize their argument, using their own notation throughout. Firstly, they expand a conjectural expression for the two-point correlation functions for the zeros of the Riemann zeta

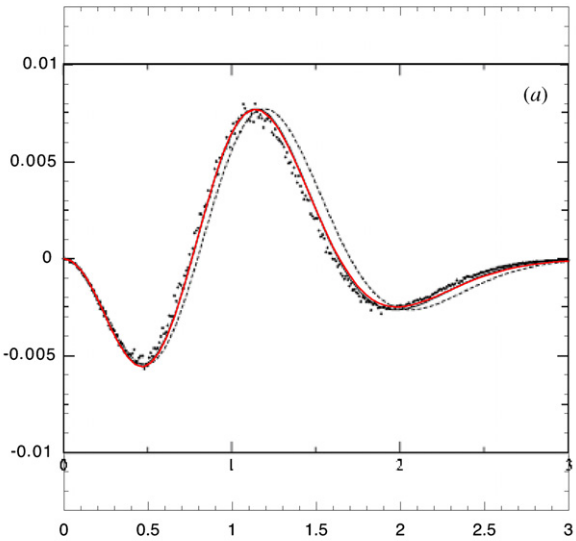


**Figure 8.** A comparison of equation (8) with figure (2) of [8]. The fit with equation (8) yields a thin curve (red online) nearly indistinguishable from the results of [8]. See the text for details.

function in inverse powers of the average density of zeros, $\bar{\rho} = 1/(2\pi)\ln(T/2\pi)$. They find that the expansion equals the leading term, which is $O(1)$, plus corrections $O(\bar{\rho}^{-2})$. Secondly, they expand the finite-$N$ expression for the two-level correlation function for unitary random matrices. They find that the expansion starts with the leading term, the same as for the Riemann zeros, plus corrections proportional to powers of $1/N^2$. By comparing the two expansions, they conclude that the two-point correlation functions of Riemann zeros and of eigenvalues of random unitary matrices, respectively, can be mapped into each other by letting $N_{\text{eff}} = \ln(T/2\pi)/(\sqrt{12\Lambda})$, as anticipated in the introduction.

Finally, they argue that this holds for all correlation functions, and thus for the spacing distribution. The correction to the leading term of $P_\zeta(s,t)$ is then found as

$$\delta p(s) = 1\big/N_{\text{eff}}^2 p_1^{(\text{CUE})}(\alpha s) \tag{15}$$

where the superscript CUE stems from the use of the circular unitary ensemble. The factor $\alpha$ in the argument of $p_1^{(\text{CUE})}$ is a function of $T$ that allows one to effectively incorporate higher-order terms. The correction $\delta p(s)$ is found numerically as the large-$N$ limit of the difference between the finite-$N$ distribution, $p^{(\text{CUE}_N)}(s)$ and the universal distribution $p_0(s)$, times $N^2$:

$$p_1^{(\text{CUE})}(s) = \lim_{N\to\infty} N^2[p^{(\text{CUE}_N)}(s) - p_0(s)]. \tag{16}$$

The type of result that they find is reproduced in figures 8 and 9, at $T = 2.504 \times 10^{15}$ and $T = 1.3066 \times 10^{22}$, respectively, together with a fit with my conjectured formula equation (8) with $t = 3.66$ and $t = 4.4$, respectively ($\alpha = 0.859$ in both cases). Bogomolny *et al* prediction essentially coincides with mine, equation (8) being much simpler and analytical.

Besides the quality of fits, one further point seems worth of consideration. Figure 7 shows that the second, third and fourth moments of $P_\zeta(s,t)$ are very close to the corresponding moments of the Wigner surmise when $T \approx 2 \times 10^{19}$. We conclude therefore that $P_\zeta(s,t) = P_2(s)$ for $t \approx 4.38$. We know that $P_2(s)$ is the exact level distribution for a $2 \times 2$ unitary matrix, so that one might expect that $N_{\text{eff}} \approx 2$ in this case, whilst $N_{\text{eff}} \approx 10$.

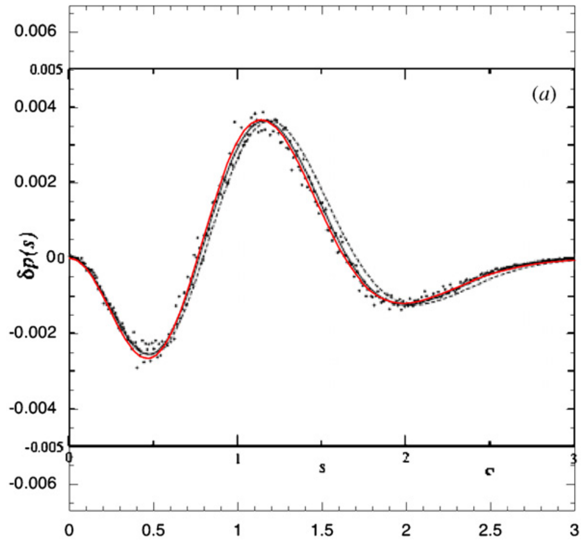


**Figure 9.** A comparison of equation (8) with figure (3) of [8]. The fit with equation (8) yields a thin curve (red online) nearly indistinguishable from the results of [8]. See the text for details.

Moreover, the $P_\zeta(s,t)$ at small $T$ cannot be interpreted as the level distributions of $N \times N$ random unitary matrices, since $P_\zeta(s,t)$ has a smaller variance than $P_2(s)$, implying that $N_{\mathrm{eff}} < 2$, which is meaningless.

The effective matrix dimension of Bogomolny *et al* seems therefore difficult to be given a precise meaning. Their approach looks inconsistent, since it describes a spacing distribution indistinguishable from that of a $2 \times 2$ matrix, as the level distribution of a much larger, $10 \times 10$, matrix.

The approach proposed here, in which the corrections to the asymptotic form of the zero spacing distribution are treated as resulting from the time evolution of an ensemble of matrices with a broken symmetry, the latter being restored as $T \to \infty$, does not suffer from similar inconsistencies.

Moreover, by relying on the results of a mean-field solution of the symmetry-restoring evolution, the present approach allows us to conjecture a very simple heuristic analytic formula, which is able to capture many fine details, including the evolution of the moments of the distribution with increasing imaginary part $T$ of the zeros.

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